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A coupled electromechanical analysis of a piezoelectric layer bonded to an elastic substrate: Part I, development of governing equations

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Abstract

This two-part contribution presents a novel and efficient method to analyze the two-dimensional (2-D) electromechanical fields of a piezoelectric layer bonded to an elastic substrate, which takes into account the fully coupled electromechanical behavior. In Part I, Hellinger–Reissner variational principle for elasticity is extended to electromechanical problems of the bimaterial, and is utilized to obtain the governing equations for the problems concerned. The 2-D electromechanical field quantities in the piezoelectric layer are expanded in the thickness-coordinate with seven one-dimensional (1-D) unknown functions. Such an expansion satisfies exactly the mechanical equilibrium equations, Gauss law, the constitutive equations, two of the three displacement–strain relations as well as one of the two electric field-electric potential relations. For the substrate the fundamental solutions of a half-plane subjected to a vertical or horizontal concentrated force on the surface are used. Two differential equations and two singular integro-differential equations of four unknown functions, the axial force, N, the moment, M, the average and the first moment of electric displacement, D_0 and D_1, as well as the associated boundary conditions have been derived rigorously from the stationary conditions of Hellinger–Reissner variational functional. In contrast to the thin film/substrate theory that ignores the interfacial normal stress the present one can predict both the interfacial shear and normal stresses, the latter one is believed to control the delamination initiation.

1. Introduction

A thin piezoelectric layer adhered on the surface of a host material or embedded in it as an actuator and/or a sensor plays an important role in the application of piezoelectric materials to smart technologies (Wang and Chen, 2000; Pal et al., 2000; Gong and Suo, 1996; Chandrasekaran et al., 2000; Kim and Jones, 1996). Two fundamental issues concerned in such an integration technology are: (1) the capability of

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converting the electric energy into the mechanical energy, and vice versa; (2) the integrity and durability of bonded smart devices or components.

The first issue is related directly to the global interaction between the piezoelectric layer and the host material. Crawley and de Luis (1987) first proposed a model to describe a deformation transfer from a piezoelectric actuator to a beam-like host material through the adhesive layer. The axial stress in the piezoceramic actuator was simply assumed to be uniform across its thickness, the adhesive layer was assumed to have a one-dimensional shear strain condition and the axial strain of the host material was assumed to be linear in the thickness direction. In order to avoid bending, the structure has two symmetrical piezoelectric actuators adhered on the top and bottom sides of the host material. Park et al. (1993) modified the model of Crawley and de Luis (1987) to the structure with a piezoelectric actuator adhered on only one side, the strain transfer rule was analyzed in the bending–extension mode and the bending–extension–distortion mode. Crawley and Anderson (1990) studied the strain transfer model of the same structure based on Euler–Bernoulli beam theory. In the paper the transverse strains of the host and of the piezoelectric actuator were both assumed to be a linear function of the thickness coordinate. Robbins and Reddy (1991) analyzed the strain transfer behavior of the structure by using the finite element method. Molyet (1999) proposed a two-dimensional (2-D) model based on the finite difference method in analyzing the beam bonded by piezoceramic pieces on both sides. Peelamedu et al. (1999) investigated the strain transfer between the actuator and the host material experimentally and numerically. The three-dimensional (3-D) finite element results were compared with the 2-D finite difference model (Molyet, 1999), the 1-D model (Park et al., 1993) and the experiments. Zhang et al. (2000, 2001a) proposed several models to deal with the conditions of plane strain, plane stress and bending in the strain transfer analysis of the beam-like structure with a piezoceramic layer adhered on one side. However, the mentioned analyses ignore the coupling between electrical fields and mechanical fields. There are a few papers concerning the coupled electromechanical analysis of piezoelectric actuator and piezoelectric sensor (Anderson and Hagood, 1994; Mitchell and Reddy, 1995; Chattopadhyay et al., 1999; Zhou et al., 2000). Zhou et al. (2000) investigated the difference between the coupled model (the two-way coupled model in the paper) and the uncoupled model (the one-way coupled model in the paper). The numerical results have shown a significant difference between the two models in some cases. Thus, a coupled piezoelectric–mechanical model would be preferred in order to deal with all the cases.

The second issue, namely the integrity and durability, is related directly to local stress fields at the sites where the mechanical and electric concentration presents. The fully coupled electromechanical analyses are needed. The earliest equations describing the mutual interactions of the piezoelectric and mechanical fields in piezoelectric materials were given by Tiersten (1969). Sosa and Pak (1990), Sosa (1992a,b) introduced the equations into the fracture analysis of piezoelectric materials and the electroelastic behavior of piezoelectric laminated structures. Most investigations to date have focused on the fracture within a single piezoelectric material (Yang and Suo, 1994; Hao et al., 1996; Zhang and Tong, 1996; Zhang et al., 1998, 2001b). Some works have been done on an interface crack between two piezoelectric materials (Suo et al., 1992; Shen et al., 1999; Herrmann et al., 2001) and between a piezoelectric material and an elastic material (Liu and Hsia, 2003; Wang and Meguid, 2000). Recently the two dimensional electromechanical singularities of piezoelectric wedges were discussed by Xu and Rajapakse (2000) and Chue and Chen (2002). The asymptotic solutions of them indicated that the stresses, electric displacements and electric fields near the apex of a wedge are proportional to r^{-s}, where r is the distance measured from the apex of the wedge. The singularity order, s, could be a complex number, leading to a physically unrealistic oscillating electroelastic field near the wedge. Most of the mentioned analyses considered an infinite body, which is somewhat not relevant geometrically to the bonded smart structures where the piezoelectric layer is of finite size and relatively thin compared with the substrate. This leaves space for one to develop an approach alternative to the mathematical wedge model, which can take into account the geometric features of the piezoelectric layer, such as the finite length and the thin thickness. In fact, Wang and Meguid (2000) investigated the behavior of a thin piezoelectric actuator attached to an infinite host structure. The uniform electrical field was assumed in piezoelectric layer. And a thin film/substrate theory was utilized, which ignores the interfacial normal stress.

When a piezoelectric layer is bonded on the surfaces of a substrate, both length and thickness of the layer are often much smaller than the size of the substrate, the host material could be treated as a half-infinite plane. In this two-part contribution, we will propose an electromechanical coupling method to analyze such problems. In this part, we devote our attention to developing the governing equations for the problems. By taking into account the geometry of the problems, the singular integro-differential equations will be derived from an extended Hellinger–Reissner variational principle.

2. Coupling problems of a piezoelectric layer bonded to an elastic substrate

Consider a piezoelectric layer of thickness, h, with the poling direction in z-axis bonded to an elastic substrate that is modeled as a half-infinite plane, Fig. 1. The Cartesian coordinate system x–y–z is used with the origin positioned at the center of the piezoelectric layer. The plane strain condition of the xz-plane is assumed. Let V and $V^{(s)}$ represent the volumes of the piezoelectric material and the substrate. The boundary value problems of the bimaterial can be stated as follows:

(1) The mechanical equilibrium equations without body force are given by

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0, \quad \text{in } V + V^{(s)}, \tag{1a}$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} = 0, \quad \text{in } V + V^{(s)}, \tag{1b}$$

where σ_x, σ_z, σ_{xz} are Cauchy stress components.

(2) Gauss law in the absence of free electric charges inside the piezoelectric material gives

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_z}{\partial z} = 0, \quad \text{in } V + V^{(s)}, \tag{2}$$

where D_x and D_z are the electric displacement components.

(3) The strain–displacement relations are

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \text{in } V + V^{(s)}, \tag{3a}$$

$$\varepsilon_z = \frac{\partial u_z}{\partial z}, \quad \text{in } V + V^{(s)}, \tag{3b}$$

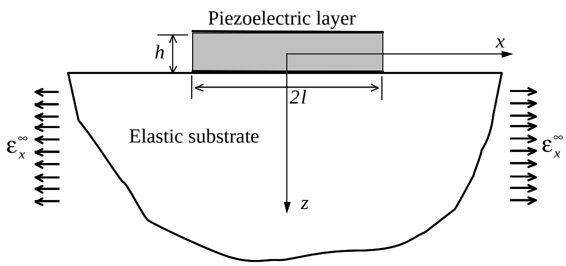

Fig. 1. A piezoelectric layer bonded to a substrate.

$$2\varepsilon_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}, \quad \text{in } V + V^{(s)}, \tag{3c}$$

where ε_x, ε_z and ε_{xz} are the infinitesimal strain components, u_x and u_z the displacement components.

(4) Under a quasi-static condition the electric field, E_x and E_z as well as the electric potential, ϕ, have the following relationship:

$$E_x = -\frac{\partial \phi}{\partial x}, \quad \text{in } V + V^{(s)}, \tag{4a}$$

$$E_z = -\frac{\partial \phi}{\partial z}, \quad \text{in } V + V^{(s)}. \tag{4b}$$

(5) The constitutive equations for the linear piezoelectric layer are given by

$$\varepsilon_x = a_{11}\sigma_x + a_{13}\sigma_z + b_{31}D_z, \tag{5a}$$

$$\varepsilon_z = a_{13}\sigma_x + a_{33}\sigma_z + b_{33}D_z, \tag{5b}$$

$$2\varepsilon_{xz} = a_{55}\sigma_{xz} + b_{15}D_x, \tag{5c}$$

$$E_x = -b_{15}\sigma_{xz} + \delta_{11}D_x, \tag{6a}$$

$$E_z = -b_{31}\sigma_x - b_{33}\sigma_z + \delta_{33}D_z, \tag{6b}$$

where a_{11}, a_{13}, a_{33} and a_{55} are the 2-D compliance components reduced from the 3-D compliance tensor s_{ijkl}; b_{15}, b_{31} and b_{33} reduced from the third-order piezoelectric tensor g_{ijk}; δ_{11} and δ_{33} reduced from the second-rank dielectric impermeability tensor β_{ij}. And the elastic stress–strain relationship of the substrate gives

$$\varepsilon_x^{(s)} = a_{11}^{(s)}\sigma_x^{(s)} + a_{13}^{(s)}\sigma_z^{(s)}, \tag{7a}$$

$$\varepsilon_z^{(s)} = a_{13}^{(s)}\sigma_x^{(s)} + a_{33}^{(s)}\sigma_z^{(s)}, \tag{7b}$$

$$2\varepsilon_{xz}^{(s)} = a_{55}^{(s)}\sigma_{xz}^{(s)}, \tag{7c}$$

where quantities with the superscript 's' belong to the substrate and those without superscript belong to the piezoelectric layer.

(6) If the two materials are perfectly bonded the displacements and mechanical tractions have to be continuous across the interface, i.e.

$$u_x^{(s)}\left(x, \frac{h}{2}\right) = u_x\left(x, \frac{h}{2}\right) = U_x(x) \quad \text{for } |x| \leqslant l, \tag{8a}$$

$$u_z^{(s)}\left(x, \frac{h}{2}\right) = u_z\left(x, \frac{h}{2}\right) = U_z(x) \quad \text{for } |x| \leqslant l, \tag{8b}$$

$$\sigma_z\left(x, \frac{h}{2}\right) = \sigma_z^{(s)}\left(x, \frac{h}{2}\right) \quad \text{for } |x| \leqslant l, \tag{9a}$$

$$\sigma_{xz}\left(x, \frac{h}{2}\right) = \sigma_{xz}^{(s)}\left(x, \frac{h}{2}\right) \quad \text{for } |x| \leqslant l. \tag{9b}$$

(7) The piezoelectric layer is free of the mechanical loading on the external surfaces, leading to the mechanical boundary conditions in terms of stress components as follows:

$$\sigma_z\left(x, -\frac{h}{2}\right) = 0, \tag{10a}$$

$$\sigma_{xz}\left(x, -\frac{h}{2}\right) = 0, \tag{10b}$$

$$\sigma_x(\pm l, z) = 0, \tag{10c}$$

$$\sigma_{xz}(\pm l, z) = 0. \tag{10d}$$

And the free surface of the substrate gives

$$\sigma_z^{(s)}\left(x, z = \frac{h}{2}\right) = 0, \quad \text{for } |x| > l, \tag{11a}$$

$$\sigma_{xz}^{(s)}\left(x, z = \frac{h}{2}\right) = 0, \quad \text{for } |x| > l. \tag{11b}$$

The elastic substrate is mechanically loaded in tension or compression via the remote axial strain, ε_x^∞, i.e.

$$\sigma_x^{(s)}(\pm\infty, z) = \frac{1}{a_{11}^{(s)}}\varepsilon_x^\infty, \tag{11c}$$

$$\sigma_{xz}^{(s)}(\pm\infty, z) = 0. \tag{11d}$$

(8) From the viewpoint of the electric boundary conditions the boundary of the piezoelectric layer can be divided into two parts, on which either the free electric charge or the electric potential is prescribed. Both the edges of the piezoelectric layer is open to air which has a dielectric constant approximately three times orders of magnitude smaller than the dielectric constants of the piezoelectric material (Sosa, 1992a). Such an assumption leads to the electrical boundary conditions given by

$$D_x(\pm l, z) = 0. \tag{12}$$

The top and bottom surfaces of the piezoelectric layer are fully covered with the electrodes of a zero thickness, and the potential on the electrodes does not vary with position, i.e.

$$\phi\left(x, -\frac{h}{2}\right) = \phi_t, \tag{13a}$$

$$\phi\left(x, \frac{h}{2}\right) = \phi_b, \tag{13b}$$

where the constant, $\phi_{bt} = \phi_b - \phi_t$, is the electric potential difference between the bottom and top electrodes. The uniform electric potentials are still unspecified, suggesting that a further boundary condition needs to be given. This can be done by one of two exclusive ways, namely by directly prescribing the voltage, V, or by giving total free charges on the surfaces, $\bar{q}$. The former one gives

$$\phi_{bt} = V. \tag{14}$$

And the latter one leads to the following mathematical expression

$$\int_{-l}^{l} D_z\left(x,-\frac{h}{2}\right)\mathrm{d}x = \int_{-l}^{l} D_z\left(x,\frac{h}{2}\right)\mathrm{d}x = \bar{q}. \tag{15}$$

Eq. (14) is exclusive from Eq. (15), and vice versa. In practice if the piezoelectric layer is used as an actuator the electric potential difference, ϕ_{bt}, is prescribed by applying a voltage, V, to the electrodes, namely Eq. (14). If the piezoelectric layer is taken as a sensor that measures the mechanical strain of the substrate an open circuit may be used without external free charges, i.e., $\bar{q} = 0$. In this case Eq. (15) should be satisfied along with $\bar{q} = 0$, and the potential difference, ϕ_{bt}, can be computed by solving the boundary value problem governed by Eqs. (1)–(13b) as well as (15), which is an indicator of the mechanical strain of the substrate.

The boundary value problem governed by the equations mentioned can be stated, alternatively, by the electromechanical variational principle given in Appendix A, which is an extension of Hellinger–Reissner variational principle for elasticity (Washizu, 1982). In the succeeding sections we will utilize the variational principle to reduce the 2-D electromechanical problem into an 1-D problem.

3. Expansion of fields in the piezoelectric layer

Assuming the x-axis normal stress has a linear variation across the thickness, it follows that

$$\sigma_x = \frac{N}{h} + \frac{12z}{h^3}M, \tag{16}$$

where N and M are the axial force and the bending moment acting on the cross-section and given by,

$$N(x) = \int_{-h/2}^{h/2} \sigma_x(x,z)\,\mathrm{d}z, \tag{17a}$$

$$M(x) = \int_{-h/2}^{h/2} z\sigma_x(x,z)\,\mathrm{d}z. \tag{17b}$$

By combining Eqs. (1a), (1b) and (16) as well as by considering the stress-free surface conditions, Eqs. (10a) and (10b), one could obtain

$$\sigma_{xz} = \frac{3(h^2-4z^2)M'}{2h^3} - \frac{(h+2z)N'}{2h}, \tag{18a}$$

$$\sigma_z = \frac{(z-h)(h+2z)^2M''}{2h^3} + \frac{(h+2z)^2N''}{8h}, \tag{18b}$$

where superscript $'$ stands for the derivative with respect to x.

Assumption that the electric displacement component, D_x, is a linear function of z-coordinate leads to

$$D_x(x,z) = D_0(x) + \frac{2z}{h}D_1(x), \tag{19}$$

where D_0 and D_1 given by

$$D_0(x) = \frac{1}{h}\int_{-h/2}^{h/2} D_x(x,z)\,\mathrm{d}z, \tag{20a}$$

$$D_1(x) = \frac{6}{h^2}\int_{-h/2}^{h/2} zD_x(x,z)\,\mathrm{d}z, \tag{20b}$$

are the through-thickness average and the first moment of the electric displacement in x-direction, respectively. By substituting Eq. (19) into Gauss law, Eq. (2), it follows

$$D_z(x,z) = -zD_0'(x) - \frac{z^2}{h}D_1'(x) + D_2(x). \tag{21}$$

Substitution of the stresses, Eqs. (16) and (18b), as well as the electric displacements, Eq. (21), into Eq. (6b), along with combination of Eqs. (4b) and (13a,13b), gives

$$\begin{aligned}\phi = {} & \frac{\phi_\mathrm{b}+\phi_\mathrm{t}}{2} + (\phi_\mathrm{b}-\phi_\mathrm{t})\frac{z}{h} - \frac{(h^2-4z^2)\delta_{33}}{8}D_0' - \frac{z(h^2-4z^2)\delta_{33}}{12h}D_1' - \frac{3b_{31}(h^2-4z^2)}{2h^3}M \\ & - \frac{(h^2-4z^2)(3h+2z)b_{33}}{48h}N'' + \frac{(h^2-4z^2)(5h^2-4z^2)b_{33}}{32h^3}M'',\end{aligned} \tag{22}$$

$$D_2 = -\frac{\phi_\mathrm{b}-\phi_\mathrm{t}}{h\delta_{33}} + \frac{hD_1'}{12} + \frac{b_{31}}{h\delta_{33}}N + \frac{hb_{33}N''}{6\delta_{33}} - \frac{b_{33}M''}{2\delta_{33}}. \tag{23}$$

Substitution of Eq. (23) into Eq. (21) leads to

$$D_z(x,z) = -\frac{\phi_\mathrm{b}-\phi_\mathrm{t}}{h\delta_{33}} - zD_0' + \left(\frac{h}{12}-\frac{z^2}{h}\right)D_1' + \frac{b_{31}}{h\delta_{33}}N + \frac{hb_{33}N''}{6\delta_{33}} - \frac{b_{33}M''}{2\delta_{33}}. \tag{24}$$

By substituting the stresses, Eqs. (16) and (18b), as well as the electric displacements, Eq. (21), into Eq. (5b), then using Eq. (3b) one could derive an expression for the z-axis displacement as follows:

$$\begin{aligned}u_z = {} & U_z + \frac{b_{33}(h-2z)}{2h\delta_{33}}(\phi_\mathrm{b}-\phi_\mathrm{t}) + \frac{(h^2-4z^2)b_{33}}{8}D_0' + \frac{z(h^2-4z^2)b_{33}}{12h}D_1' \\ & - \frac{(b_{31}b_{33}+a_{13}\delta_{33})(h-2z)}{2h\delta_{33}}N - \frac{3a_{13}(h^2-4z^2)}{2h^3}M \\ & - \frac{(h-2z)\left(4b_{33}^2h^2 + a_{33}\delta_{33}(7h^2+8hz+4z^2)\right)}{48h\delta_{33}}N'' \\ & + \frac{(h-2z)\left(8b_{33}^2h^3 + a_{33}\delta_{33}(13h^3+10h^2z-4hz^2-8z^3)\right)}{32h^3\delta_{33}}M''\end{aligned} \tag{25}$$

where $U_z(x)$ is the z-axis displacement of the bottom surface of the piezoelectric layer. Similarly, an appropriate combination of the stress, Eq. (18a), the electric displacement, Eq. (19), as well as Eqs. (5c), (3c) and (25), leads to

$$\begin{aligned}u_x = {} & U_x + \left(\frac{h}{2}-z\right)U_z' - \frac{b_{15}(h-2z)}{2}D_0 - b_{15}\left(\frac{h}{4}-\frac{z^2}{h}\right)D_1 + \frac{(h-2z)^2(h+z)b_{33}}{24}D_0'' \\ & + \frac{(h^2-4z^2)^2b_{33}}{192h}D_1'' - \frac{(b_{31}b_{33}(h-2z) + (a_{13}-3a_{55})\delta_{33}h - 2(a_{13}+a_{55})\delta_{33}z)(h-2z)}{8h\delta_{33}}N' \\ & - \frac{(a_{13}+a_{55})(h-2z)^2(h+z)}{2h^3}M' - \frac{(h-2z)^2\left(8b_{33}^2h^2 + a_{33}\delta_{33}(17h^2+12hz+4z^2)\right)}{384h\delta_{33}}N''' \\ & + \frac{(h-2z)^2\left(10b_{33}^2h^3 + a_{33}\delta_{33}(18h^3+7h^2z-4hz^2-4z^3)\right)}{160h^3\delta_{33}}M'''\end{aligned} \tag{26}$$

where $U_x(x)$ is the interface displacement in x-direction. In conclusion, the 2-D fields of stresses, displacements, electric displacements and electric potentials in the piezoelectric layer have been expressed in terms of seven unknown quantities, N, M, U_x, U_z, D_0, D_1 and ϕ_{bt}. The last one is a constant while the others are functions of x-coordinate only. Such an expansion of the mechanical and electric fields satisfies the mechanical equilibrium equations (1a) and (1b), Gauss law, Eq. (2), the displacement–strain relations, Eqs. (3b) and (3c), the potential-electric field relation, Eq. (4b), as well as the constitutive equations. The remaining two field equations (3a) and (4a) will be replaced by the stationary conditions of Hellinger–Reissner variational functional given in Appendix A, along with substitution of the expanded fields.

4. Mechanical fields of the substrate

In order to express the mechanical fields of the substrate in terms of the interface displacements, U_x and U_z, the exact solutions of the half-plane subjected to a concentrated force on the surface are utilized. A horizontal concentrated force of unit magnitude at the point, ξ, on the surface will produce a relative x-axis displacement of point, x, with respect to point x_0, on the surface, given by

$$G_x(x,\xi) = \frac{2(1-\nu^{(s)^2})}{\pi Y^{(s)}} \ln\left|\frac{x_0-\xi}{x-\xi}\right|, \tag{27}$$

and the vanishing relative vertical displacement. $Y^{(s)}$ and $\nu^{(s)}$ are Young's modulus and Poisson's ratio of the substrate, respectively. When the surface of the half plane is subjected to a vertical concentrated force of unit magnitude at the point, ξ, the relative vertical displacement of point, x, with respect to point x_0, on the surface could be expressed as:

$$G_z(x,\xi) = \frac{2(1-\nu^{(s)^2})}{\pi Y^{(s)}} \ln\left|\frac{x_0-\xi}{x-\xi}\right|, \tag{28}$$

and the relative x-axis displacement is zero. Equations (27) and (28) valid for the plane strain problem can apply to the plane stress case by replacing $Y^{(s)}/(1-\nu^{(s)^2})$ with $Y^{(s)}$.

Assume that the surface tractions over the portion $(-l, l)$ are denoted by $p_z(x)$ and $p_x(x)$, it follows:

$$p_x = -\sigma_{xz}^{(s)}\left(x, \frac{h}{2}\right), \tag{29a}$$

$$p_z = -\sigma_z^{(s)}\left(x, \frac{h}{2}\right). \tag{29b}$$

To produce the interface displacements, U_x and U_z, which are the relative displacements of point, x, with respect to the point x_0, the tractions on the surface of the half-plane, $p_z(x)$ and $p_x(x)$, as well as the remote strain have to satisfy, i.e.

$$U_x = (x-x_0)\varepsilon_x^\infty + \frac{2(1-\nu^{(s)^2})}{\pi Y^{(s)}} \int_{-l}^{l} p_x(\xi) \ln\left|\frac{x_0-\xi}{x-\xi}\right| \mathrm{d}\xi, \tag{30}$$

$$U_z = \frac{2(1-\nu^{(s)^2})}{\pi Y^{(s)}} \int_{-l}^{l} p_z(\xi) \ln\left|\frac{x_0-\xi}{x-\xi}\right| \mathrm{d}\xi. \tag{31}$$

5. Governing equations

Now all electric quantities in the piezoelectric layer and the mechanical fields in both the piezoelectric material and the substrate have been expressed in terms of the 1-D unknown functions, N, M, U_x, U_z, D_0, D_1

and ϕ_{bt}. The governing equations for them could be obtained by substituting the expanded electromechanical fields into the extended Hellinger–Reissner functional, Π_R, given in Appendix A, and by making use of the stationary conditions. Since the solutions for stresses and displacements in the half-plane produced by interface displacements, U_x and U_z, are exact the variation of the extended Hellinger–Reissner functional associated with the substrate is absolutely zero. Therefore, the extended Hellinger–Reissner functional, Eq. (A.12), along with the substitution of the electromechanical fields given by Eqs. (16),(18a), (18b), (19), (24)–(26), (29a)–(31) leads to

$$\delta\Pi_R = \delta\Pi_V + \delta\Pi_{SM} + \delta\Pi_{SE}, \tag{32}$$

in which three terms are associated with the contributions, respectively, of volume, mechanical boundaries and electric boundaries of the piezoelectric material, i.e.

$$\begin{aligned}\delta\Pi_V = &\int_V \left(\frac{\partial u_x}{\partial x} - (a_{11}\sigma_x + a_{13}\sigma_z + b_{31}D_z)\right)\left(\frac{1}{h}\delta N + \frac{12z}{h^3}\delta M\right)\mathrm{d}V \\ &- \int_V \left((-b_{15}\sigma_{xz} + \delta_{11}D_x) + \frac{\partial\phi}{\partial x}\right)\left(\frac{1}{h}\delta D_0 + \frac{12z}{h^3}\delta D_1\right)\mathrm{d}V,\end{aligned} \tag{33}$$

$$\begin{aligned}\delta\Pi_{SM} = &-\int_{-l}^{l}\left(\sigma_z\left(x,\frac{h}{2}\right) + p_z(x)\right)\delta U_z \mathrm{d}x - \int_{-l}^{l}\left(\sigma_{xz}\left(x,\frac{h}{2}\right) + p_x(x)\right)\delta U_x \mathrm{d}x \\ &+ \int_{-h/2}^{h/2}\sigma_x(l,z)\delta u_x(l,z)\,\mathrm{d}z + \int_{-h/2}^{h/2}\sigma_{xz}(l,z)\delta u_z(l,z)\,\mathrm{d}z - \int_{-h/2}^{h/2}\sigma_x(-l,z)\delta u_x(-l,z)\,\mathrm{d}z \\ &- \int_{-h/2}^{h/2}\sigma_{xz}(-l,z)\delta u_z(-l,z)\,\mathrm{d}z,\end{aligned} \tag{34}$$

$$\delta\Pi_{SE} = (\phi_{bt} - V)\int_{-l}^{l}\delta D_z\left(x,\frac{h}{2}\right)\mathrm{d}x - \int_{-h/2}^{h/2}D_x(l,z)\delta\phi(l,z)\,\mathrm{d}z + \int_{-h/2}^{h/2}D_x(-l,z)\delta\phi(-l,z)\,\mathrm{d}z \tag{35a}$$

or

$$\delta\Pi_{SE} = -\left(\int_{-l}^{l}D_z\left(x,\frac{h}{2}\right)\mathrm{d}x - \bar{q}\right)\delta\phi_{bt} - \int_{-h/2}^{h/2}D_x(l,z)\delta\phi(l,z)\,\mathrm{d}z + \int_{-h/2}^{h/2}D_x(-l,z)\delta\phi(-l,z)\,\mathrm{d}z. \tag{35b}$$

Eqs. (35a) and (35b) are exclusive, representing the boundary conditions, respectively, given by prescribing the electric potential and the total free charge on both the top and bottom surfaces. The independent quantities subject to variation in the functional (32)–(35) are seven quantities, N, M, U_x, U_z, D_0, D_1 and ϕ_{bt}. The explicit expressions for the first variation of the functional, Eqs. (33)–(35b), are given in Appendix B. The stationary condition $\delta\Pi_R = 0$ along with Eqs. (32)–(34) gives

$$\int_{-h/2}^{h/2}\left(\frac{\partial u_x}{\partial x} - (a_{11}\sigma_x + a_{13}\sigma_z + b_{31}D_z)\right)\mathrm{d}z = 0 \tag{36}$$

$$\int_{-h/2}^{h/2}\left(\frac{\partial u_x}{\partial x} - (a_{11}\sigma_x + a_{13}\sigma_z + b_{31}D_z)\right)z\,\mathrm{d}z = 0, \tag{37}$$

$$\int_{-h/2}^{h/2}\left((-b_{15}\sigma_{xz} + \delta_{11}D_x) + \frac{\partial\phi}{\partial x}\right)\mathrm{d}z = 0, \tag{38}$$

$$\int_{-h/2}^{h/2}\left((-b_{15}\sigma_{xz}+\delta_{11}D_x)+\frac{\partial\phi}{\partial x}\right)z\,\mathrm{d}z=0. \tag{39}$$

$$p_x=N', \tag{40}$$

$$p_z=M''-\frac{h}{2}N''. \tag{41}$$

Substitution of the expanded electromechanical fields into equations (36)–(39) yields the following four equations,

$$\begin{aligned}U_x'=&-\frac{\phi_{\mathrm{bt}}b_{31}}{h\delta_{33}}+\frac{6a_{11}}{h^2}M+\frac{(b_{31}^2+a_{11}\delta_{33})}{h\delta_{33}}N-\frac{(5b_{31}b_{33}+(12a_{13}+a_{55})\delta_{33})}{10\delta_{33}}M''\\&+\frac{h(b_{31}b_{33}+4a_{13}\delta_{33}-a_{55}\delta_{33})}{12\delta_{33}}N''+\frac{h^2(35b_{33}^2+44a_{33}\delta_{33})}{840\delta_{33}}M''''-\frac{h^3(5b_{33}^2+6a_{33}\delta_{33})}{360\delta_{33}}N''''\\&-\frac{b_{31}h}{2}D_0'+\frac{b_{15}h}{6}D_1'+\frac{b_{33}h^3}{120}D_0'''-\frac{b_{33}h^3}{360}D_1'''\end{aligned} \tag{42}$$

$$\begin{aligned}U_z''=&-\frac{12a_{11}}{h^3}M+\frac{6(2a_{13}+a_{55})}{5h}M''+\frac{(b_{31}b_{33}-a_{55}\delta_{33})}{2\delta_{33}}N''-\frac{h(35b_{33}^2+52a_{33}\delta_{33})}{140\delta_{33}}M''''\\&+\frac{h^2(5b_{33}^2+8a_{33}\delta_{33})}{60\delta_{33}}N''''+(b_{15}+b_{31})D_0'-\frac{b_{33}h^2}{10}D_0''',\end{aligned} \tag{43}$$

$$\delta_{11}hD_0-\frac{h^3\delta_{33}}{12}D_0''-(b_{15}+b_{31})M'+\frac{b_{15}h}{2}N'+\frac{b_{33}h^2}{10}M'''-\frac{b_{33}h^3}{24}N'''=0, \tag{44}$$

$$-120\delta_{11}D_1+2h^2\delta_{33}D_1''-60b_{15}N'+h^2b_{33}N'''=0. \tag{45}$$

From Eqs. (30), (31), (40) and (41) follows

$$U_x=(x-x_0)\varepsilon_x^\infty+\frac{2(1-\nu^{(s)^2})}{\pi Y^{(s)}}\int_{-l}^{l}N'(\xi)\ln\left|\frac{x_0-\xi}{x-\xi}\right|\mathrm{d}\xi, \tag{46}$$

$$U_z=\frac{2(1-\nu^{(s)^2})}{\pi Y^{(s)}}\int_{-l}^{l}\left(M''(\xi)-\frac{h}{2}N''(\xi)\right)\ln\left|\frac{x_0-\xi}{x-\xi}\right|\mathrm{d}\xi. \tag{47}$$

By eliminating U_x and U_z from Eqs. (42) and (43) and Eqs. (46) and (47), one obtain two integro-differential equations as follows,

$$\begin{aligned}&T_{11}N''''+T_{12}M''''+T_{13}N''+T_{14}M''+T_{15}D_1'+T_{16}D_0'+T_{17}N+T_{18}M+\int_{-l}^{l}\frac{N'(\xi)}{x-\xi}\mathrm{d}\xi\\&\quad=\frac{\pi Y^{(s)}}{2(1-\nu^{(s)^2})}\left(\varepsilon_x^\infty+\frac{\phi_{\mathrm{bt}}b_{31}}{h\delta_{33}}\right),\end{aligned} \tag{48}$$

$$T_{21}N''''+T_{22}M''''+T_{23}N''+T_{24}M''+T_{25}D_0'+T_{26}M+\int_{-l}^{l}\frac{hN''(\xi)-2M''(\xi)}{(x-\xi)^2}\mathrm{d}\xi=0. \tag{49}$$

where coefficient constants T_{kl} are listed in Appendix C. Now, Eqs. (44) and (45) and (48) and (49) are the governing integro-differential equations in terms of force, moment and electric displacements for the piezoelectric layer bonded to an elastic substrate. From the stationary conditions $\delta \Pi_R = 0$ with explicit expression of Eqs. (34) and (35), one obtains the mechanical boundary conditions

$$N(\pm l) = 0, \quad M(\pm l) = 0, \tag{50a,b}$$

$$N'(\pm l) = 0, \quad M'(\pm l) = 0, \tag{51a,b}$$

and the electric boundary conditions on the edges

$$D_0(\pm l) = 0, \quad D_1(\pm l) = 0. \tag{52a,b}$$

The electric boundary condition on the top and bottom surfaces coated with electrodes can be obtained as,

$$\phi_{\mathrm{bt}} = V, \tag{53}$$

for a prescribed voltage, V, or

$$\phi_{\mathrm{bt}} = \frac{b_{31}}{2l} \int_{-l}^{l} N(x)\,\mathrm{d}x - \frac{h\delta_{33}}{2l} \bar{q}, \tag{54}$$

for a prescribed free charge on the surfaces, $\bar{q}$.

6. Conclusions

We have considered electromechanical coupling problems that a piezoelectric layer bonded to elastic substrate. Two differential equations and two singular integro-differential equations of four unknown functions, the axial force, N, the moment, M, the average and the first moment of electric displacement, D_0 and D_1, as well as the associated boundary conditions have been derived rigorously from the stationary conditions of Hellinger–Reissner variational functional. The 2-D electromechanical field quantities in piezoelectric layer satisfies exactly the equilibrium equations and Gauss law, the constitutive equations, two of the three displacement–strain relations as well as one of the two electric field-electric potential relations. Remaining two equations are satisfied by the stationary conditions of the variational functional. Values of all electromechanical quantities everywhere in the layer and the substrate can be computed from the four variables. In contrast to the thin film/substrate theory that ignores the interfacial normal stress the present one can predict both the interfacial shear and normal stresses, the latter one is believed to control the delamination initiation.

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Appendix A. Hellinger–Reissner variational principle for piezoelectric bimaterials

Consider a bimaterial system which consists of two dissimilar piezoelectric materials of volumes, $V^{(1)}$ and $V^{(2)}$, perfectly bonded together. Let S denote the external boundary surface of the bimaterial, and $S^{(12)}$ the interface. The standard Einstein notation is used with summation convention applied to the repeated indices. The boundary value problems of the bimaterial can be stated as follows:

(1) The mechanical equilibrium equations without body force are given by

$$\sigma_{ji,j} = 0 \quad \text{in } V^{(1)} + V^{(2)}, \tag{A.1}$$

where σ_{ij} is the Cauchy stress tensor.

(2) Gauss law in the absence of free electric charge inside the materials gives

$$D_{i,i} = 0 \quad \text{in } V^{(1)} + V^{(2)}, \tag{A.2}$$

where D_i is the electric displacement vector.

(3) The strain–displacement relations are

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } V^{(1)} + V^{(2)}, \tag{A.3}$$

where ε_{ij} is the infinitesimal strain tensor, and u_i the displacement vector.

(4) Under a quasi-static condition the electric field, E_i, and the electric potential, ϕ, have the following relationship:

$$E_i = -\phi_{,i} \quad \text{in } V^{(1)} + V^{(2)}. \tag{A.4}$$

(5) The constitutive equations for linear piezoelectricity are given by

$$\varepsilon_{ij} = s_{ijkl}\sigma_{kl} + g_{kij}D_k \quad \text{in } V^{(1)} + V^{(2)}, \tag{A.5a}$$

$$E_k = -g_{ijk}\sigma_{ij} + \beta_{kl}D_l \quad \text{in } V^{(1)} + V^{(2)}, \tag{A.5b}$$

where s_{ijkl} is the fourth-order tensor of elastic compliance measured at zero electric displacement, g_{ijk} the third-order piezoelectric tensor and β_{ik} the second-rank dielectric impermeability tensor measured at zero stress.

(6) From the viewpoint of the mechanical boundary conditions the external boundary, S, can be divided into two parts, S_σ and S_u, on which tractions and displacements are prescribed by $\bar{T}_i$ and $\bar{u}_i$, respectively, i.e.

$$\sigma_{ij}n_i = \bar{T}_j \quad \text{on } S_\sigma, \tag{A.6a}$$

$$u_i = \bar{u}_i \quad \text{on } S_u, \tag{A.6b}$$

where n_i are the direction cosines of the outward unit vector normal to the boundary. From the viewpoint of the electric boundary conditions the external boundary, S, can be divided into two parts, S_D and S_ϕ, on which free electric charge and electric potential are prescribed by $\bar{q}$ and $\bar{\phi}$, respectively, i.e.

$$D_i n_i = \bar{q} \quad \text{on } S_D, \tag{A.7a}$$

$$\phi = \bar{\phi} \quad \text{on } S_\phi. \tag{A.7b}$$

(7) If the two materials are perfectly bonded the mechanical tractions and displacements have to be continuous across the interface, i.e.

$$(\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)})n_i^{(1)} = 0 \text{ on } S^{(12)}, \tag{A.8a}$$

$$u_i^{(1)} = u_i^{(2)}, \text{ on } S^{(12)}, \tag{A.8b}$$

where $n_i^{(1)}$ are the direction cosines of the unit vector normal to the interface pointing from the material "1" to the material "2". In absence of the free charge on the interface there is no jump in the electric displacement across the interface, i.e.

$$(D_i^{(1)} - D_i^{(2)})n_i^{(1)} = 0 \quad \text{on } S^{(12)}. \tag{A.9a}$$

The electric potential has to be continuous across the interface,

$$\phi^{(1)} = \phi^{(2)} \quad \text{on } S^{(12)}. \tag{A.9b}$$

The exact solution for a boundary value problem of the jointed two piezoelectric materials has to satisfy all Eqs. (A.1)–(A.9b). Alternatively, the problem could be stated by a variational principle. By taking the constitutive equations (A.5a) and (A.5b), the interface continuity conditions of the displacement and the electric potential, Eqs. (A.8b) and (A.9b), as subsidiary conditions, Hellinger–Reissner variational principle for elasticity (Washizu, 1982) can be extended to the jointed piezoelectric materials by introducing the following functional including electro-mechanical coupling

$$\begin{aligned} \Pi_R = {} & \sum_k \int_{V^{(k)}} \left(\tfrac{1}{2}\sigma_{ij}(u_{i,j}+u_{j,i}) - D_i\phi_{,i} - B(\sigma_{ij}, D_i)\right)\mathrm{d}V - \int_{S_\sigma} \bar{T}_i u_i \,\mathrm{d}S - \int_{S_u} (u_i - \bar{u}_i) n_j \sigma_{ji}\,\mathrm{d}S \\ & + \int_{S_D} \bar{q}\phi\,\mathrm{d}S + \int_{S_\phi} (\phi - \bar{\phi}) n_i D_i \,\mathrm{d}S - \int_{S^{(12)}} (n_j^{(1)}\sigma_{ji}^{(1)} + n_j^{(2)}\sigma_{ji}^{(2)}) u_i^{(1)}\,\mathrm{d}S + \int_{S^{(12)}} (n_j^{(1)} D_j^{(1)} + n_j^{(2)} D_j^{(2)})\phi^{(1)}\,\mathrm{d}S, \end{aligned} \tag{A.10}$$

where

$$B = \tfrac{1}{2}\left(\sigma_{ij}s_{ijkl}\sigma_{kl} + D_i\beta_{ij}D_j - \sigma_{ij}g_{ijk}D_k + D_k g_{kij}\sigma_{ij}\right), \tag{A.11}$$

is complementary energy density. The independent quantities subject to variation in the functional (A.10) are stress, σ_{ij}, displacements, u_i, electric displacements, D_i, and electric potential, ϕ. By taking variation of the functional (A.10) with respect to those quantities one obtains

$$\begin{aligned} \delta\Pi_R = {} & \sum_k \int_{V^{(k)}} \left[\left(\frac{1}{2}(u_{i,j}+u_{j,i}) - \frac{\partial B}{\partial \sigma_{ij}}\right)\delta\sigma_{ji} - \left(\frac{\partial B}{\partial D_i} + \phi_{,i}\right)\delta D_i - \sigma_{ij,i}\delta u_j + D_{i,i}\delta\phi\right]\mathrm{d}V \\ & + \int_{S_\sigma} (n_j\sigma_{ji} - \bar{T}_i)\delta u_i\,\mathrm{d}S - \int_{S_u} (u_i - \bar{u}_i) n_j \delta\sigma_{ji}\,\mathrm{d}S - \int_{S_D} (n_i D_i - \bar{q})\delta\phi\,\mathrm{d}S + \int_{S_\phi} (\phi - \bar{\phi}) n_i \delta D_i\,\mathrm{d}S \\ & - \int_{S^{(12)}} n_j^{(1)}(\sigma_{ji}^{(1)} - \sigma_{ji}^{(2)})\delta u_i^{(1)}\mathrm{d}S + \int_{S^{(12)}} n_j^{(1)}(D_j^{(1)} - D_j^{(2)})\delta\phi^{(1)}\mathrm{d}S \end{aligned} \tag{A.12}$$

and the stationary condition $\delta\Pi_R = 0$ lead to Eqs. (A.1), (A.2), (A.6a)–(A.8a), (A.9a) as well as

$$\tfrac{1}{2}(u_{i,j}+u_{j,i}) = s_{ijkl}\sigma_{kl} + g_{kij}D_k \quad \text{in } V^{(1)} + V^{(2)}, \tag{A.13a}$$

$$-\phi_{,k} = -g_{ijk}\sigma_{ij} + \beta_{kl}D_l \quad \text{in } V^{(1)} + V^{(2)}. \tag{A.13b}$$

Eqs. (A.13a) and (A.13b) are equivalent to the strain–displacement relations given by Eq. (A.3) as well as the electric field-electric potential relations given by Eq. (A.4) if one takes the constitutive equations (A.5a) and (A.5b) as subsidiary conditions.

Therefore, the boundary value problems governed by Eqs. (A.1)–(A.9b) can be stated by the stationary conditions of the functional (A.10) along with the constitutive equations (A.5a) and (A.5b) as well as the interface continuity conditions of the displacement and the electric potential, Eqs. (A.8b) and (A.9b), as subsidiary conditions.

Appendix B. Integrals in the variational functional

$$\int_{-h/2}^{h/2}\sigma_x(\pm l,z)\delta u_x(\pm l,z)\,\mathrm{d}z=\Bigg[N\delta U_x+\frac{b_{15}}{2}(2M-hN)\delta D_0-\frac{hb_{15}}{6}N\delta D_1-\frac{1}{2}(2M-hN)\delta U_z'$$
$$+\frac{3(b_{31}b_{33}+\delta_{33}a_{13}-\delta_{33}a_{55})M-h(b_{31}b_{33}+\delta_{33}a_{13}-2\delta_{33}a_{55})N}{6\delta_{33}}\delta N'$$
$$+\frac{(a_{13}+a_{55})(12M-5hN)}{10h}\delta M'-\frac{b_{33}h^2}{120}(12M-5hN)\delta D_0''+\frac{b_{33}h^3}{360}N\delta D_1''$$
$$+\frac{3h^2(5b_{33}^2+8\delta_{33}a_{33})M-h^3(5b_{33}^2+9\delta_{33}a_{33})N}{6\delta_{33}}\delta N'''$$
$$+\frac{-3h(35b_{33}^2+52\delta_{33}a_{33})M+7h^2(5b_{33}^2+8\delta_{33}a_{33})N}{420\delta_{33}}\delta M'''\Bigg]_{x=\pm l},$$

$$\int_{-h/2}^{h/2}\sigma_{xz}(\pm l,z)\delta u_z(\pm l,z)\,\mathrm{d}z=\Bigg[\frac{b_{33}(3M'-hN')\delta\phi_{\mathrm{bt}}}{6\delta_{33}}+\frac{1}{2}(2M'-hN')\delta U_z$$
$$-\frac{(b_{31}b_{33}+\delta_{33}a_{13})(3M'-hN')}{6\delta_{33}}\delta N+\frac{a_{13}(-12M+5hN)}{10h}\delta M$$
$$+\frac{h^2b_{33}}{120}(12M'-5hN')\delta D_0'-\frac{h^3b_{33}}{360}N'\delta D_1'$$
$$+\frac{-3h^2(5b_{33}^2+8\delta_{33}a_{33})M+h^3(5b_{33}^2+9\delta_{33}a_{33})N}{180\delta_{33}}\delta N''$$
$$-\frac{-3h(35b_{33}^2+52\delta_{33}a_{33})M'+7h^2(5b_{33}^2+8\delta_{33}a_{33})N'}{420\delta_{33}}\delta M''\Bigg]_{x=\pm l},$$

$$\int_{-h/2}^{h/2}D_x(\pm l,z)\delta\phi(\pm l,z)\,\mathrm{d}z=\Bigg[-\frac{1}{6}D_1h\delta\phi_{\mathrm{bt}}+\frac{1}{2}hD_0(\delta\phi_{\mathrm{b}}+\delta\phi_{\mathrm{t}})-b_{31}D_0\delta M-\frac{h^3\delta_{33}}{12}D_0\delta D_0'$$
$$-\frac{h^3\delta_{33}}{180}D_1\delta D_1'-\frac{h^3b_{33}(15D_0+D_1)}{360}\delta N''+\frac{b_{33}h^2D_0}{10}\delta M''\Bigg]_{x=\pm l},$$

$$(\phi_{\mathrm{bt}}-V)\int_{-l}^{l}\delta D_z\left(x,\frac{h}{2}\right)\mathrm{d}x=\frac{2l}{h\delta_{33}}(\phi_{\mathrm{bt}}-V)\delta\phi_{\mathrm{bt}}+(\phi_{\mathrm{bt}}-V)\frac{b_{33}}{h\delta_{33}}\int_{-l}^{l}\delta N(x)\,\mathrm{d}x$$
$$+(\phi_{\mathrm{bt}}-V)\left(\left[-\frac{h}{2}\delta D_0-\frac{h}{6}\delta D_1+\frac{hb_{33}}{6\delta_{33}}\delta N'-\frac{hb_{33}}{2\delta_{33}}\delta M'\right]_{x=l}\right.$$
$$\left.-\left[-\frac{h}{2}\delta D_0-\frac{h}{6}\delta D_1+\frac{hb_{33}}{6\delta_{33}}\delta N'-\frac{hb_{33}}{2\delta_{33}}\delta M'\right]_{x=-l}\right),$$

$$\delta\phi_{\mathrm{bt}}\int_{-l}^{l}D_z\left(x,\frac{h}{2}\right)\mathrm{d}x=\frac{2l}{h\delta_{33}}\delta\phi_{\mathrm{bt}}\phi_{\mathrm{bt}}+\delta\phi_{\mathrm{bt}}\frac{b_{33}}{h\delta_{33}}\int_{-l}^{l}N(x)\,\mathrm{d}x+\delta\phi_{\mathrm{bt}}\left(\left[-\frac{h}{2}D_0-\frac{h}{6}D_1+\frac{hb_{33}}{6\delta_{33}}N'-\frac{hb_{33}}{2\delta_{33}}M'\right]_{x=l}\right.$$
$$\left.-\left[-\frac{h}{2}D_0-\frac{h}{6}D_1+\frac{hb_{33}}{6\delta_{33}}N'-\frac{hb_{33}}{2\delta_{33}}M'\right]_{x=-l}\right).$$

Appendix C. Coefficient constants in the governing equations

$$T_{11} = -\frac{\pi Y^{(s)}}{2(1 - {v^{(s)}}^2)}\left(\frac{h^3 a_{33}}{60} + \frac{h^3 b_{33}^2}{60\delta_{33}}\right),$$

$$T_{12} = \frac{\pi Y^{(s)}}{2(1 - {v^{(s)}}^2)}\left(\frac{11h^2 a_{33}}{210} + \frac{217h^2 b_{33}^2}{4200\delta_{33}}\right),$$

$$T_{13} = \frac{\pi Y^{(s)}}{2(1 - {v^{(s)}}^2)}\left(\frac{h(20a_{13} - 5a_{55})}{60} + \frac{hb_{33}(-2b_{15} + 5b_{31})}{60\delta_{33}}\right),$$

$$T_{14} = -\frac{\pi Y^{(s)}}{2(1 - {v^{(s)}}^2)}\left(\frac{(12a_{13} + a_{55})}{10} + \frac{(b_{15} + 6b_{31})b_{33}}{10\delta_{33}}\right),$$

$$T_{15} = \frac{\pi Y^{(s)}}{2(1 - {v^{(s)}}^2)}\frac{(-b_{33}\delta_{11} + b_{15}\delta_{33})h}{6\delta_{33}},$$

$$T_{16} = \frac{\pi Y^{(s)}}{2(1 - {v^{(s)}}^2)}\frac{(b_{33}\delta_{11} - 5b_{31}\delta_{33})h}{10\delta_{33}},$$

$$T_{17} = \frac{\pi Y^{(s)}}{2(1 - {v^{(s)}}^2)}\left(\frac{a_{11}}{h} + \frac{b_{31}^2}{h\delta_{33}}\right),$$

$$T_{18} = \frac{3a_{11}}{h^2}\frac{\pi Y^{(s)}}{(1 - {v^{(s)}}^2)},$$

$$T_{21} = \frac{\pi Y^{(s)}}{(1 - {v^{(s)}}^2)}\frac{2h^2(b_{33}^2 + a_{33}\delta_{33})}{15\delta_{33}},$$

$$T_{22} = -\frac{\pi Y^{(s)}}{(1 - {v^{(s)}}^2)}\left(\frac{13ha_{33}}{35} + \frac{259hb_{33}^2}{700\delta_{33}}\right),$$

$$T_{23} = -\frac{\pi Y^{(s)}}{(1 - {v^{(s)}}^2)}\left(\frac{a_{55}}{2} + \frac{(6b_{15} - 5b_{31})b_{33}}{10\delta_{33}}\right),$$

$$T_{24} = \frac{\pi Y^{(s)}}{(1 - {v^{(s)}}^2)}\left(\frac{6(2a_{13} + a_{55})}{5h} + \frac{6(b_{15} + b_{31})b_{33}}{5h\delta_{33}}\right),$$

$$T_{25} = \frac{\pi Y^{(s)}}{(1 - {v^{(s)}}^2)}\left(b_{15} + b_{31} - \frac{6b_{33}\delta_{11}}{5\delta_{33}}\right),$$

$$T_{26} = -\frac{12a_{11}}{h^3}\frac{\pi Y^{(s)}}{(1 - {v^{(s)}}^2)}.$$

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